

ON MULTILINEAR POLYNOMIALS IN FOUR VARIABLES EVALUATED ON MATRICES

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ABSTRACT. Let K be an algebraically closed field of characteristic 0 and let $M_n(K)$, $n \geq 3$, be the matrix ring over K . We will show that the image of any multilinear polynomial in four variables evaluated on $M_n(K)$ contains all matrices of trace 0.

1. INTRODUCTION

In 1936 Shoda [9] proved that over a field of characteristic 0, every matrix of trace 0 can be represented as a commutator $AB - BA$ of two matrices. In 1957 Albert and Muckenhoupt [1] extended the result to all fields. An old open question (Problem 1.98 in the Dniester Notebook, Fourth Edition, communicated by Lvov, also attributed to Kaplansky, see [6]) asks: "Let f be a multilinear polynomial over a field K . Is the set of values of f on the matrix algebra $M_n(K)$ a vector space?" There were few developments in this problem until recently when Kanel-Belov, Malev and Rowen solved the problem for 2×2 matrices [6] and made important progress towards the 3×3 case [7].

Shoda's theorem can be reformulated in the form that the set of values of the polynomial $f(x, y) = xy - yx$ on the algebra of matrices contains all matrices of trace 0. In 2013 Shoda's result was generalized by Mesyan [8] for multilinear polynomials of degree 3 and by Špenko [10] for Lie polynomials of degree ≤ 4 .

Throughout the paper, we will use $[M_n(K), M_n(K)]$ to denote the K -subspace of $M_n(K)$ consisting of the matrices of trace zero and $f(M_n(K))$ to denote the set of values of a polynomial f on $M_n(K)$. By a multilinear polynomial we will understand the polynomial in noncommutative variables and linear in each variable. Our main goal is to prove the following result:

Theorem 1. *Let $n \geq 3$ be an integer, K an algebraically closed field of characteristic 0, and $f \in K\langle x_1, x_2, x_3, x_4 \rangle$ any nonzero multilinear polynomial. Then $[M_n(K), M_n(K)] \subseteq f(M_n(K))$.*

The condition on $n \geq 3$ is necessary as, for instance, the theorem fails for $n = 2$ and $f(x_1, x_2, x_3, x_4) = [x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2]$ which is a famous example of a central polynomial.

Our theorem confirms the following conjecture due to Mesyan [8, Conjecture 11] for the case $m = 4$:

Conjecture. Let K be a field, $n \geq 2$ and $m \geq 1$ integers, and $f(x_1, \dots, x_m)$ a nonzero multilinear polynomial in $K\langle x_1, \dots, x_m \rangle$. If $n \geq m - 1$, then $[M_n(K), M_n(K)] \subseteq f(M_n(K))$.

This problem remains open for $m > 4$.

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2. THE RESULTS

We will start with several auxiliary results. By $e_{i,j}$ we denote a standard matrix unit, that is an $n \times n$ matrix with 1 in the i 'th row and j 'th column and zeros elsewhere.

Lemma 2. *Let K be a field and let $a_{i,j} \in K$ be any elements with $\sum_{i=1}^n a_{i,i} = 0$. Let $A = \sum_{i=1}^{n-1} e_{i,i+1} \in M_n(K)$. Then there exists a $B \in M_n(K)$ such that*

$$(1) \quad [A, B] = \sum_{i=1}^n a_{i,i} e_{i,i} + \sum_{i=1}^{n-1} a_{i,i+1} e_{i,i+1}.$$

Proof. We will be looking for a matrix B in the form $\sum_{i=1}^n b_{i,i} e_{i,i} + \sum_{i=1}^{n-1} b_{i+1,i} e_{i+1,i}$. Then

$$(2) \quad [A, B] = b_{2,1} e_{1,1} + \sum_{i=2}^{n-1} (b_{i+1,i} - b_{i,i-1}) e_{i,i} - b_{n,n-1} e_{n,n} + \sum_{i=1}^{n-1} (b_{i+1,i+1} - b_{i,i}) e_{i,i+1}.$$

Comparing (1) and (2), we arrive at two systems of linear equations in indeterminates $b_{i,j}$'s.

$$(3) \quad \begin{aligned} b_{2,1} &= a_{1,1} \\ b_{3,2} - b_{2,1} &= a_{2,2} \\ &\vdots \\ b_{n,n-1} - b_{n-1,n-2} &= a_{n-1,n-1} \\ b_{n,n-1} &= \sum_{i=1}^{n-1} a_{i,i}, \end{aligned}$$

and

$$(4) \quad \begin{aligned} b_{2,2} - b_{1,1} &= a_{1,2} \\ b_{3,3} - b_{2,2} &= a_{2,3} \\ &\vdots \\ b_{n,n} - b_{n-1,n-1} &= a_{n-1,n}. \end{aligned}$$

Clearly, both systems have solutions. For example, the system (3) has solutions:

$$\begin{aligned} b_{2,1} &= a_{1,1} \\ b_{3,2} &= a_{1,1} + a_{2,2} \\ &\vdots \\ b_{n,n-1} &= a_{1,1} + a_{2,2} + \dots + a_{n-1,n-1}, \end{aligned}$$

and the system (4) has solutions:

$$\begin{aligned} b_{1,1} &= 0 \\ b_{2,2} &= a_{1,2} \\ b_{3,3} &= a_{1,2} + a_{2,3} \\ b_{4,4} &= a_{1,2} + a_{2,3} + a_{3,4} \\ &\vdots \\ b_{n,n} &= a_{1,2} + a_{2,3} + \dots + a_{n-1,n}. \end{aligned}$$

This completes the lemma. □

We will need the following well known facts about the images of multilinear polynomials that can be found, for example, in [8, Lemma 6 and Corollary 8].

Remark 3. Let K be a field, m a positive integer, and

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_m} a_{\sigma} x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(m)} \in K\langle x_1, \dots, x_m \rangle$$

a multilinear polynomial, where S_m is the permutation group on m elements.

(i) If $\sum_{\sigma \in S_m} a_{\sigma} \neq 0$, then $f(M_n(K)) = M_n(K)$.

(ii) For any invertible matrix $A \in M_n(K)$ and any matrices $B_1, \dots, B_m \in M_n(K)$ we have

$$Af(B_1, \dots, B_m)A^{-1} = f(AB_1A^{-1}, \dots, AB_mA^{-1}).$$

We are ready to prove our next lemma.

Lemma 4. Let K be an algebraically closed field of characteristic 0. Let λ be an element of K and $\lambda \neq -1$. Then for every matrix $D \in M_n(K)$ of trace 0 with $n \geq 3$, there exist $A, B, C \in M_n(K)$ such that $D = [A, B][A, C] + \lambda[A, C][A, B]$.

Proof. Any matrix with coefficients in an algebraically closed field is similar to its Jordan canonical form, so by Remark 3 it is enough to show that every matrix D of trace 0 in the form $\sum_{i=1}^n d_{i,i} e_{i,i} + \sum_{i=1}^{n-1} d_{i,i+1} e_{i,i+1}$ can be represented as $[A, B][A, C] + \lambda[A, C][A, B]$. We will consider two cases.

Case 1: Suppose $\lambda \neq \frac{1}{n-1}$. Then by Lemma 2 if we let $A = \sum_{i=1}^{n-1} e_{i,i+1}$, then there exist $B, C \in M_n(K)$ such that $[A, B] = \sum_{i=1}^{n-1} e_{i,i} - (n-1)e_{n,n}$ and $[A, C] = \sum_{i=1}^n b_{i,i} e_{i,i} + \sum_{i=1}^{n-1} b_{i,i+1} e_{i,i+1}$ where $b_{n,n} = -\sum_{i=1}^{n-1} b_{i,i}$. Then we obtain

$$\begin{aligned} [A, B][A, C] + \lambda[A, C][A, B] = \\ (\lambda + 1) \sum_{i=1}^{n-1} b_{i,i} e_{i,i} - (n-1)(\lambda + 1)b_{n,n} e_{n,n} + (\lambda + 1) \sum_{i=1}^{n-2} b_{i,i+1} e_{i,i+1} + (1 - (n-1)\lambda)b_{n-1,n} e_{n-1,n}. \end{aligned}$$

Like in Lemma 2, we arrive at two systems of linear equations in indeterminates $b_{i,j}$'s.

$$\begin{aligned} (\lambda + 1)b_{1,1} &= d_{1,1} \\ (\lambda + 1)b_{2,2} &= d_{2,2} \\ &\vdots \\ (\lambda + 1)b_{n-1,n-1} &= d_{n-1,n-1} \\ -(n-1)(\lambda + 1)b_{n,n} &= d_{n,n}, \end{aligned}$$

(we included the last equation just for consistency, it is actually the sum of all previous equations multiplied by -1), and

$$\begin{aligned} (\lambda + 1)b_{1,2} &= d_{1,2} \\ (\lambda + 1)b_{2,3} &= d_{2,3} \\ &\vdots \\ (\lambda + 1)b_{n-2,n-1} &= d_{n-2,n-1} \\ (1 - (n-1)\lambda)b_{n-1,n} &= d_{n-1,n}. \end{aligned}$$

Since $\lambda \neq -1$ and $\lambda \neq \frac{1}{n-1}$, obviously both systems have solutions.

Case 2: Suppose $\lambda = \frac{1}{n-1}$. Then we take $[A, B] = \sum_{i=1}^{n-2} e_{i,i} + 2e_{n-1,n-1} - ne_{n,n}$ and $[A, C] = \sum_{i=1}^n b_{i,i}e_{i,i} + \sum_{i=1}^{n-1} b_{i,i+1}e_{i,i+1}$ where $b_{n,n} = -\sum_{i=1}^{n-1} b_{i,i}$. Then we obtain

$$\begin{aligned} [A, B][A, C] + \lambda[A, C][A, B] &= (\lambda + 1) \sum_{i=1}^{n-2} b_{i,i}e_{i,i} + 2(\lambda + 1)b_{n-1,n-1}e_{n-1,n-1} - n(\lambda + 1)b_{n,n}e_{n,n} \\ &\quad + (\lambda + 1) \sum_{i=1}^{n-3} b_{i,i+1}e_{i,i+1} + (1 + 2\lambda)b_{n-2,n-1}e_{n-2,n-1} + (2 - n\lambda)b_{n-1,n}e_{n-1,n}. \end{aligned}$$

Again we arrive at two systems of linear equations

$$\begin{aligned} (\lambda + 1)b_{1,1} &= d_{1,1} \\ (\lambda + 1)b_{2,2} &= d_{2,2} \\ &\vdots \\ (\lambda + 1)b_{n-2,n-2} &= d_{n-2,n-2} \\ 2(\lambda + 1)b_{n-1,n-1} &= d_{n-1,n-1} \\ -n(\lambda + 1)b_{n,n} &= d_{n,n}, \end{aligned}$$

and

$$\begin{aligned} (\lambda + 1)b_{1,2} &= d_{1,2} \\ (\lambda + 1)b_{2,3} &= d_{2,3} \\ &\vdots \\ (\lambda + 1)b_{n-3,n-2} &= d_{n-3,n-2} \\ (2\lambda + 1)b_{n-2,n-1} &= d_{n-2,n-1} \\ (2 - n\lambda)b_{n-1,n} &= d_{n-1,n}. \end{aligned}$$

Since $\lambda = \frac{1}{n-1}$, both systems have solutions.

Therefore, we can get the desired representation and the proof is complete. \square

The next result follows from the proof of [10, Lemma 7.4]:

Lemma 5. *Let K be an algebraically closed field of characteristic 0. Then for every matrix $D \in M_n(K)$ of trace 0 with $n \geq 3$, there exist $A, B, C \in M_n(K)$ such that $D = [A, B][A, C] - [A, C][A, B]$.*

We continue with

Lemma 6. *Let $n \geq 3$ be an integer. Let K be a field of characteristic 0 and let $a_{ij} \in K$ be any elements with $\sum_{i=1}^n a_{i,i} = 0$. Then there exist $A, B, C \in M_n(K)$ such that $[A, [[A, B], [A, C]]] = \sum_{i=1}^n a_{i,i}e_{i,i} + \sum_{i=1}^{n-1} a_{i,i+1}e_{i,i+1}$.*

Proof. Let $A \in M_n(K)$ be a matrix of the form $\sum_{i=1}^{n-1} e_{i,i+1}$. Then let $B, C \in M_n(K)$ such that

$$(5) \quad B = \sum_{i=1}^n b_{i,i}e_{i,i} + \sum_{i=1}^{n-1} b_{i+1,i}e_{i+1,i}$$

with $\sum_{i=1}^n b_{i,i} = 0$, and $C = \sum_{i=3}^n (i-2)e_{i,i-2}$. Then $[A, C] = \sum_{i=1}^{n-2} e_{i+1,i} - (n-2)e_{n,n-1}$ and $[A, B] = \sum_{i=1}^{n-1} (b_{i+1,i+1} - b_{i,i})e_{i,i+1} + b_{2,1}e_{1,1} + \sum_{i=2}^{n-1} (b_{i+1,i} - b_{i,i-1})e_{i,i} - b_{n,n-1}e_{n,n}$. Note that for any $c_{i,j}$ with

$\sum_{i=1}^n c_{i,i} = 0$, we can get $[A, B]$ in the form $[A, B] = \sum_{i=1}^n c_{i,i} e_{i,i} + \sum_{i=1}^{n-1} c_{i,i+1} e_{i,i+1}$. Indeed, the two systems of linear equations in variables $b_{i,j}$

$$\begin{aligned} b_{2,2} - b_{1,1} &= c_{1,2} \\ b_{3,3} - b_{2,2} &= c_{2,3} \\ &\vdots \\ b_{n,n} - b_{n-1,n-1} &= c_{n-1,n}, \end{aligned}$$

and

$$\begin{aligned} b_{2,1} &= c_{1,1} \\ b_{3,2} - b_{2,1} &= c_{2,2} \\ b_{4,3} - b_{3,2} &= c_{3,3} \\ &\vdots \\ b_{n,n-1} - b_{n-1,n-2} &= c_{n-1,n-1} \\ -b_{n,n-1} &= c_{n,n} \end{aligned}$$

have solutions satisfying $\sum_{i=1}^n b_{i,i} = 0$. Then for $[A, B] = \sum_{i=1}^n c_{i,i} e_{i,i} + \sum_{i=1}^{n-1} c_{i,i+1} e_{i,i+1}$ and $[A, C] = \sum_{i=1}^{n-2} e_{i+1,i} - (n-2)e_{n,n-1}$ we get

$$\begin{aligned} [[A, B], [A, C]] &= c_{1,2} e_{1,1} + \sum_{i=2}^{n-2} (c_{i,i+1} - c_{i-1,i}) e_{i,i} + (-(n-2)c_{n-1,n} - c_{n-2,n-1}) e_{n-1,n-1} \\ &\quad + (n-2)c_{n-1,n} e_{n,n} + \sum_{i=1}^{n-2} (c_{i+1,i+1} - c_{i,i}) e_{i+1,i} - (n-2)(c_{n,n} - c_{n-1,n-1}) e_{n,n-1}. \end{aligned}$$

Observe that for any $d_{i,j}$ with $\sum_{i=1}^n d_{i,i} = 0$, the systems of linear equations in variables $c_{i,j}$

$$\begin{aligned} c_{2,2} - c_{1,1} &= d_{2,1} \\ c_{3,3} - c_{2,2} &= d_{3,2} \\ &\vdots \\ c_{n-1,n-1} - c_{n-2,n-2} &= d_{n-1,n-2} \\ -(n-2)(c_{n,n} - c_{n-1,n-1}) &= d_{n,n-1}, \end{aligned}$$

and

$$\begin{aligned} c_{1,2} &= d_{1,1} \\ c_{2,3} - c_{1,2} &= d_{2,2} \\ c_{3,4} - c_{2,3} &= d_{3,3} \\ &\vdots \\ c_{n-2,n-1} - c_{n-3,n-2} &= d_{n-2,n-2} \\ -(n-2)c_{n-1,n} - c_{n-2,n-1} &= d_{n-1,n-1} \\ (n-2)c_{n-1,n} &= d_{n,n} \end{aligned}$$

have solutions satisfying $\sum_{i=1}^n c_{i,i} = 0$. Now we have $D = [[A, B], [A, C]] = \sum_{i=1}^n d_{i,i} e_{i,i} + \sum_{i=1}^{n-1} d_{i+1,i} e_{i+1,i}$ with $\sum_{i=1}^n d_{i,i} = 0$. Observe that the matrix D is of the same form as matrix B in equation (5) so $[A, D]$ can be represented in the form $\sum_{i=1}^n a_{i,i} e_{i,i} + \sum_{i=1}^{n-1} a_{i,i+1} e_{i,i+1}$. \square

Before we start the proof of our main result, we need to handle one special case.

Proposition 7. *Let $n \geq 3$ be an integer and let K be an algebraically closed field of characteristic 0. If*

$$f(x_1, x_2, x_3, x_4) = [x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2] + [x_2, x_3][x_1, x_4] + [x_1, x_4][x_2, x_3] - [x_1, x_3][x_2, x_4] - [x_2, x_4][x_1, x_3],$$

then $[M_n(K), M_n(K)] \subseteq f(M_n(K))$.

Proof. Let $A, B, C \in M_n(K)$. Then

$$\begin{aligned} f(A, A^2, B, C) &= [A, A^2][B, C] + [B, C][A, A^2] + [A^2, B][A, C] \\ &\quad + [A, C][A^2, B] - [A, B][A^2, C] - [A^2, C][A, B] \\ &= [A^2, B][A, C] + [A, C][A^2, B] - [A, B][A^2, C] \\ &\quad - [A^2, C][A, B] \\ &= [A, [[A, B], [A, C]]]. \end{aligned}$$

By Lemma 6, there exist $A, B, C \in M_n(K)$ such that $f(A, A^2, B, C)$ can be represented as $\sum_{i=1}^n a_{i,i} e_{i,i} + \sum_{i=1}^{n-1} a_{i,i+1} e_{i,i+1}$ with $\sum_{i=1}^n a_{i,i} = 0$. This includes the Jordan canonical forms of all matrices with trace zero. By Remark 3, we obtain $[M_n(K), M_n(K)] \subseteq f(M_n(K))$. \square

We are ready to prove our main result.

Proof of Theorem 1. We can write any multilinear polynomial $f \in K\langle x_1, x_2, x_3, x_4 \rangle$ in the form

$$f(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} a_{\sigma} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)},$$

where S_4 is the permutation group on four elements. If $\sum_{\sigma \in S_4} a_{\sigma} \neq 0$, then by Remark 3, $f(M_n(K))$ contains all matrices, so in particular $[M_n(K), M_n(K)] \subseteq f(M_n(K))$.

Therefore, we can assume $\sum_{\sigma \in S_4} a_{\sigma} = 0$.

If the partial derivative of f with respect to some x_i is nonzero, then we can set x_i equal to the identity matrix 1 and the situation reduces to the three variable case covered in [8, Theorem 13].

Therefore we can assume all partial derivatives of f are zero, that is

$$f(1, x_2, x_3, x_4) = f(x_1, 1, x_3, x_4) = f(x_1, x_2, 1, x_4) = f(x_1, x_2, x_3, 1) = 0.$$

By Falk's Theorem [3] (see also [5, Theorem 1.1] for the English version) any such polynomial is a product of (iterated) commutators. It means that our polynomial can be written in the form

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= L(x_1, x_2, x_3, x_4) + c_{1234}[x_1, x_2][x_3, x_4] + c_{1324}[x_1, x_3][x_2, x_4] + c_{1423}[x_1, x_4][x_2, x_3] \\ &\quad + c_{2314}[x_2, x_3][x_1, x_4] + c_{2413}[x_2, x_4][x_1, x_3] + c_{3412}[x_3, x_4][x_1, x_2], \end{aligned}$$

where $L(x_1, x_2, x_3, x_4)$ is a Lie polynomial in four variables. According to [4, Theorem 3.1], every such polynomial can be written as follows:

$$\begin{aligned} L(x_1, x_2, x_3, x_4) &= z_1[[x_2, x_1], x_3], x_4] + z_2[[x_3, x_1], x_2], x_4] + z_3[[x_4, x_1], x_2], x_3] \\ &\quad + z_4[[x_4, x_1], [x_3, x_2]] + z_5[[x_4, x_2], [x_3, x_1]] + z_6[[x_4, x_3], [x_2, x_1]]. \end{aligned}$$

Observe that the last three terms can be written as a linear combination of products of commutators. Thus without loss of generality we may assume that $f(x_1, x_2, x_3, x_4)$ is of the form

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = & z_1 [[[x_2, x_1], x_3], x_4] + z_2 [[[x_3, x_1], x_2], x_4] + z_3 [[[x_4, x_1], x_2], x_3] \\ & + c_{1234}[x_1, x_2][x_3, x_4] + c_{1324}[x_1, x_3][x_2, x_4] + c_{1423}[x_1, x_4][x_2, x_3] \\ & + c_{2314}[x_2, x_3][x_1, x_4] + c_{2413}[x_2, x_4][x_1, x_3] + c_{3412}[x_3, x_4][x_1, x_2]. \end{aligned}$$

Suppose that for some $i = 1, 2, 3$, $z_i \neq 0$. Say, let $z_1 \neq 0$. Arguing as in the proof of [10, Lemma 7.4], we take $x_1 = x_3 = x_4 = S$, where S is a diagonal matrix with distinct diagonal entries. By [2, Lemma 1.2], $f(S, x_2, S, S)$ consists of all matrices with only zeros on the main diagonal. By [2, Proposition 1.8], every matrix of trace zeros is similar to a matrix with only zero on the main diagonal, so by Remark 3, we obtain $[M_n(K), M_n(K)] \subseteq f(M_n(K))$. The cases when $z_2 \neq 0$ or $z_3 \neq 0$ can be treated similarly. Now we can assume that $z_1 = z_2 = z_3 = 0$ and f is of the form

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = & c_{1234}[x_1, x_2][x_3, x_4] + c_{1324}[x_1, x_3][x_2, x_4] + c_{1423}[x_1, x_4][x_2, x_3] \\ & + c_{2314}[x_2, x_3][x_1, x_4] + c_{2413}[x_2, x_4][x_1, x_3] + c_{3412}[x_3, x_4][x_1, x_2]. \end{aligned}$$

Consider two cases. Case 1: Assume $c_{1234} = c_{2314} = c_{3412} = c_{1423} = -c_{1324} = -c_{2413}$. Then

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = & c_{1234}([x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2] + [x_2, x_3][x_1, x_4] + [x_1, x_4][x_2, x_3] \\ & - [x_1, x_3][x_2, x_4] - [x_2, x_4][x_1, x_3]). \end{aligned}$$

Applying Proposition 7, we can conclude $[M_n(K), M_n(K)] \subseteq f(M_n(K))$.

Case 2: Assume at least one of the following $c_{1234} = c_{2314} = c_{3412} = c_{1423} = -c_{1324} = -c_{2413}$ does not hold. For any $A, B, C \in M_n(K)$, at least one of the following expressions is not zero:

$$f(A, A, B, C) = (c_{1324} + c_{2314})[A, B][A, C] + (c_{1423} + c_{2413})[A, C][A, B],$$

$$f(A, B, A, C) = (c_{1234} - c_{2314})[A, B][A, C] + (c_{3412} - c_{1423})[A, C][A, B],$$

$$f(A, B, C, A) = (-c_{1234} - c_{2413})[A, B][A, C] + (-c_{1324} - c_{3412})[A, C][A, B],$$

$$f(B, A, A, C) = (-c_{1234} - c_{1324})[A, B][A, C] + (-c_{2413} - c_{3412})[A, C][A, B],$$

$$f(B, A, C, A) = (-c_{1423} + c_{1234})[A, B][A, C] + (c_{3412} - c_{2314})[A, C][A, B],$$

$$f(B, C, A, A) = (c_{1324} + c_{1423})[A, B][A, C] + (c_{2314} + c_{2413})[A, C][A, B].$$

The problem is now reduced to the polynomial $[A, B][A, C] + \lambda[A, C][A, B]$. If $\lambda \neq -1$, then by Lemma 4, the image of f contains all matrices of trace zero and our theorem is complete. If $\lambda = -1$, then the desired result follows from Lemma 5. The proof is complete.

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